

HKT MANIFOLDS WITH HOLONOMY $SL(n, H)$

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ABSTRACT. We show that on an HKT manifold the holonomy of the Obata connection is contained in $SL(n, \mathbb{H})$ if and only if the Lee form is an exact one form. As an application, we show compact HKT manifolds with holomorphically trivial canonical bundle which are not balanced. A simple criterion for non-existence of HKT metric on hypercomplex manifold is given in terms of the Ricci-type tensors of the Obata connection.

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1. INTRODUCTION

We recall that an HKT structure on an hyperhermitian manifold is a linear connection with totally skew-symmetric torsion preserving the hyperhermitian structure. If the torsion three form is closed (resp. trace-free) the HKT structure is called strong (resp. balanced). If the torsion vanishes one has hyperKähler manifold. HKT structures are present in many branches of theoretical and mathematical physics. For instance, they appear on supersymmetric sigma models with Wess-Zumino term [14, 22, 21] as well as in supergravity theories [37, 17, 34].

There are known some geometrical and topological properties of HKT manifold. A simple characterization of the existence of HKT structure is obtained in terms of the intrinsic torsion of an $Sp(n)Sp(1)$ structure [30]. It is shown in [31, 4] that, as in the hyperKähler case, locally any HKT metric admits an HKT potential. A version of Hodge theory has been given in [39] discovering the remarkable analogy between the de Rham complex of a Kähler manifold and the Dolbeault complex on HKT manifold.

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A special attention is paid for HKT manifold with holomorphically trivial canonical bundle with respect to any complex structure from the hypercomplex family. HKT manifold with holomorphic volume form appear as solutions of the gravitino and dilatino Killing spinor equations in dimension $4n$ with more than two supersymmetries preserved [37]. It was observed by M. Verbitsky in [40] that, in the compact case, the latter condition can be expressed in terms of the Obata connection [33] which is the unique torsion-free connection preserving the hypercomplex structure. M. Verbitsky proved in [40] that a compact HKT manifold has holomorphically trivial canonical bundle exactly when the holonomy of the Obata connection is a subgroup of the special quaternionic linear group $SL(n, \mathbb{H})$ which leads to invent into consideration the notion of $SL(n, \mathbb{H})$ manifolds. The group $SL(n, \mathbb{H})$ is one of possible holonomy groups of a torsion-free linear connection in the Merkulov-Schwachhöfer list [31]. The $SL(n, \mathbb{H})$ manifolds were studied in [2, 41] discovering that the quaternionic Dolbeault complex can be identified with a part of the de Rham complex. For a hypercomplex manifold with holomorphically trivial canonical bundle admitting an HKT metric, a version of Hodge theory constructed in [39] leads to the fact established in [40] that a compact hypercomplex manifold with holomorphically trivial canonical bundle is $SL(n, \mathbb{H})$ manifold if it admits an HKT-structure. A. Swann constructed in [38] compact simply connected $SL(n, \mathbb{H})$ manifolds which do not admit any HKT structure which, in particular, shows the existence of compact hypercomplex manifolds with holomorphically trivial canonical bundle with no HKT metric.

Special attention deserve balanced HKT metrics. It was shown in [42] that a balanced HKT manifold is an $SL(n, \mathbb{H})$ manifold. Balanced HKT metrics seem to be the quaternionic analogue of the Calabi-Yau metrics defined in terms of quaternionic Monge-Ampere equation [2, 42]. A quaternionic version of the famous Calabi-Yau theorem conjectured in [2, 42] states that on a compact HKT manifold with $SL(n, \mathbb{H})$ -holonomy of the Obata connection there exists a balanced HKT metric and if it exists it is unique in its cohomology class.

The main purpose of this note is to find precise simple condition on an HKT manifold to have holomorphically trivial canonical bundle. We show in Theorem 2.2 below that the necessary and sufficient condition an HKT manifold to have $SL(n, \mathbb{H})$ -holonomy of the Obata connection is that a certain trace of the torsion three form, called the Lee form, is an exact one form.

Examples of compact HKT manifold with holomorphically trivial canonical bundle are all nilmanifolds with an abelian hypercomplex structure since they are balanced HKT spaces [5]. Compact examples of such spaces which are not nilmanifold were presented in [6] which are again balanced. There are known compact simply connected HKT manifolds with holomorphically trivial canonical bundle constructed by A. Swann [38] via the twist construction.

Remark 1.1. *Applying Theorem 2.2 to the explicit examples of HKT manifold presented in [6], Example 6.1 and Example 6.2] we obtain that these HKT manifold are $SL(2, \mathbb{H})$ -manifold since the corresponding Lee form is exact. This provides non-balanced compact HKT manifolds with holomorphically trivial canonical bundle.*

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2. HYPERCOMPLEX, $SL(n, \mathbb{H})$ AND HKT MANIFOLDS

An almost hypercomplex structure on a $4n$ -dimensional manifold M is a triple $H = (J_s), s = 1, 2, 3$, of almost complex structures $J_s : TM \rightarrow TM$ satisfying the quaternionic identities

$J_s^2 = -id_{TM}$ and $J_1 J_2 = -J_2 J_1 = J_3$. When each J_s is a complex structure, H is said to be a hypercomplex structure on M and (M, H) is called a hypercomplex manifold. A hyperhermitian metric is a Riemannian metric g which is Hermitian with respect to each almost complex structure in H , $g(J_s \cdot, J_s \cdot) = g(\cdot, \cdot)$, $s = 1, 2, 3$. A hyperhermitian manifold (M, H, g) consists of a hypercomplex structure H and a compatible hyperhermitian metric g . The fundamental 2-forms of a hyperhermitian manifold (M, H, g) are globally defined by $F_s(\cdot, \cdot) = g(\cdot, J_s \cdot)$, $s = 1, 2, 3$. If the three fundamental 2-forms are closed we have *hyperKähler manifold*.

2.1. HKT manifolds. A hyperhermitian manifold (M, H, g) is called *hyperkähler with torsion (HKT-manifold)* if there exists a linear connection preserving the hyperhermitian structure and having totally skew-symmetric torsion, or equivalently, if the following condition is satisfied [22]

$$(2.1) \quad \begin{aligned} J_1 dF_1 &= J_2 dF_2 = J_3 dF_3, \quad \text{where} \\ J_s dF_s(X, Y, Z) &= -dF_s(J_s X, J_s Y, J_s Z), \quad s = 1, 2, 3. \end{aligned}$$

Each F_s is an $(1,1)$ -form with respect to J_s . One can also associate a non-degenerate complex 2-form $\Omega_i = F_j + \sqrt{-1}F_k$, where $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$. The 2-form Ω_i is of type $(2,0)$ with respect to the complex structure J_i . If the 2-form Ω_i is closed then the manifold is hyperKähler. The hyperkähler with torsion condition is equivalent to the condition $\partial_{J_i} \Omega = 0$ [18]. It was observed in [30] that the condition (2.1) implies that the almost hypercomplex structure is hypercomplex, thus reducing the definition of HKT-manifold as an almost hyperhermitian manifold satisfying (2.1).

2.2. Supersymmetry and HKT manifolds. The notion of HKT-manifold was introduced in physics by Howe and Papadopoulos [22] in connection with $(4,0)$ supersymmetric sigma models with non vanishing Wess-Zumino term. HKT manifolds are also connected with the supersymmetric string backgrounds [37]. The bosonic fields of the ten-dimensional supergravity which arises as low energy effective theory of the heterotic string are the spacetime metric g , the NS three-form field strength H , the dilaton ϕ and the gauge connection A with curvature F^A . One considers the connection $\nabla = \nabla^g + \frac{1}{2}H$, where ∇^g is the Levi-Civita connection of the Riemannian metric g . The connection ∇ preserves the metric, $\nabla g = 0$ and has totally skew-symmetric torsion $T = H$.

A heterotic geometry will preserve supersymmetry if and only if there exists at least one Majorana-Weyl spinor ϵ such that the supersymmetry variations of the fermionic fields vanish, i.e. the following Killing-spinor equations hold [37]

$$(2.2) \quad \delta_\lambda = \nabla \epsilon = 0; \quad \delta_\Psi = (d\phi - \frac{1}{2}H) \cdot \epsilon = 0; \quad \delta_\xi = F^A \cdot \epsilon = 0,$$

where λ, Ψ, ξ are the gravitino, the dilatino and the gaugino fields, respectively and \cdot means Clifford action of forms on spinors.

The whole Strominger system has an additional equation, called anomaly cancellation (see [37]), expressing dH in terms of the first Pontryagin form of the instanton connection A and a certain connection on the tangent bundle (which turns out to be of instanton type [25]). We note that the first compact solutions to the whole Strominger system with non-trivial fields in dimension six and non-constant dilaton were constructed in [29] (see also [12, 13, 7]) and the first explicit compact examples with constant dilaton are constructed in [9].

We briefly explain here the geometry arising from the first two equations in (2.2). The first equation in (2.2) leads, in even dimensions $2n$, to the existence of an $SU(n)$ -structure, i.e. the existence of an almost complex structure J hermitian compatible with the metric g and a non-vanishing complex (with respect to J) volume form which are preserved by the metric connection with totally skew-symmetric torsion ∇ , hence the holonomy group of ∇ is contained in $SU(n)$.

The second equation in (2.2) forces the almost complex structure to be integrable and the non-vanishing complex volume form to be a holomorphic volume form, i.e. complex manifold with holomorphically trivial canonical bundle [37]. It turns out that in the case of compact non-Kähler solution to the first two equations in (2.2) the holomorphic $(n,0)$ form is unique [3, 27]. Strominger shows [37] that, in the complex case, the torsion three form of ∇ is unique given by

$$(2.3) \quad T = JdF_J,$$

where F_J is the fundamental two form of (g, J) . The equation (2.3) combined with the classical result that a metric connection is completely determined by its torsion implies that the connection ∇ preserving the hermitian structure (J, g) and having totally skew-symmetric torsion always exists and it is unique. Thus one has the notion of Kähler manifold with torsion (KT manifold), (M, J, g, ∇) . It follows from (2.3) that the torsion three form is of type $(1,2)+(2,1)$ since the almost complex structure is integrable. The connection ∇ with torsion three form preserving a hermitian structure was independently used by Bismut [8] to prove a local index formula for the Dolbeault operator when the manifold is not Kähler and it was sometimes called the Bismut connection.

If the torsion three form is closed, $dT = 0$ then the KT-manifold is called *strong KT manifold*. These spaces are connected with the supersymmetric string background of type IIA, IIB (see eg [16] and references therein). It is easy to see from (2.3) that the strong KT-condition $dT = 0$ is equivalent to the condition $\partial\bar{\partial}F_J = 0$. Such hermitian spaces are also known as pluriclosed hermitian manifolds and a Ricci-type flow is investigated in [36]. If the trace of the exterior derivative of the torsion is zero, $g(dT, F_J) = 0$, (equivalently the trace of $\partial\bar{\partial}F_J$ is zero, $g(\partial\bar{\partial}F_J, F_J) = 0$), we have the notion of almost strong KT manifold and vanishing theorems on compact almost strong KT manifolds are presented in [28, 27].

Note that for almost hermitian manifold the existence of a connection with totally skew-symmetric torsion preserving the almost hermitian structure is obstructive. It was observed in [11] that the obstruction is encoded into the properties of the Nijenhuis tensor N_J , namely, such a connection exists on an almost hermitian manifold exactly when the Nijenhuis tensor is a three form, $N_J(X, Y, Z) = g(N_J(X, Y), Z) = -g(N_J(X, Z), Y)$. In this case the connection is unique and its torsion three form T is given by [11]

$$(2.4) \quad T = JdF_J + N_J.$$

On an almost Hermitian manifold the $(3,0)+(0,3)$ part dF_J^- of the exterior derivative of the fundamental 2-form is determined by the Nijenhuis tensor [15] and if the Nijenhuis tensor is a three form then the formula takes the form [11] $JdF_J^- = -\frac{3}{4}N_J$. An important special case is the Nearly Kähler manifold [19] which is characterized by the condition $JdF_J = JdF_J^- = -\frac{3}{4}N_J$.

In the case of dimension $4n$, the existence of more than two parallel spinors in the first equation of (2.2) leads to the existence of an $Sp(n)$ -structure, i.e. the existence of an almost hyperhermitian structure (g, H) which is preserved by the metric connection with totally skew-symmetric torsion ∇ , hence the holonomy group of ∇ is contained in $Sp(n)$ (see also the recent paper [23]). Since for each $J_s \in H$ the connection ∇ is unique, we obtain from (2.4)

Proposition 2.1. *An almost hyperhermitian manifold (M, g, H) admits a connection ∇ with skew-symmetric torsion preserving the hyperhermitian structure if and only if the Nijenhuis tensors $N_{J_1}, N_{J_2}, N_{J_3}$ are three forms and the following conditions hold*

$$J_1dF_{J_1} + N_{J_1} = J_2dF_{J_2} + N_{J_2} = J_3dF_{J_3} + N_{J_3}.$$

If the hypercomplex structure is integrable we have an HKT manifold. However, an almost hyperhermitian structure which is consisted of three Nearly Kähler structures admits hyperhermitian connection with torsion three form if and only if it is hyper Kähler. Indeed,

the nearly Kähler conditions $J_s dF_{J_s} = -\frac{3}{4}N_{J_s}$, $s = 1, 2, 3$ together with Proposition 2.1 imply $J_1 dF_{J_1} = J_2 dF_{J_2} = J_3 dF_{J_3}$ and the already mentioned result in [30] gives that the hypercomplex structure is integrable and we have hyper Kähler manifold.

The second equation in (2.2) forces that each almost complex structure $J_s \in H$ is integrable [37], i.e. we have a hyperhermitian manifold which has to satisfy (2.1). Thus, we have an HKT manifold with torsion three form given by

$$(2.5) \quad T = J_1 dF_{J_1} = J_2 dF_{J_2} = J_3 dF_{J_3}.$$

Following [37], the first two equations in (2.2) imply that the HKT manifold admits a non-degenerate holomorphic $(2n, 0)$ form with respect to any complex structure $J_s \in H$, i.e. the canonical bundle of the corresponding KT manifolds $(M, g, J_s \in H, \nabla)$ is holomorphically trivial. It was observed by M. Verbitsky in [40] that, in the compact case, the latter condition may be expressed in terms of the Obata connection of a hypercomplex manifold leading to invent into consideration the notion of $SL(n, \mathbb{H})$ manifolds.

2.3. $SL(n, \mathbb{H})$ manifolds. It was shown by Obata in [33] that a hypercomplex manifold (M, H) admits a unique torsion-free connection preserving the complex structures J_s , $s = 1, 2, 3$. We shall call this connection *the Obata connection* and denote it with ∇^{ob} . The Obata connection is uniquely determined by the conditions $\nabla^{ob} J_1 = \nabla^{ob} J_2 = \nabla^{ob} J_3 = T^{ob} = 0$. The converse is also true, namely if an almost hypercomplex manifold admits a torsion-free connection preserving the almost hypercomplex structure then it is a hypercomplex manifold.

The holonomy of the Obata connection, $Hol(\nabla^{ob})$ is contained in the general quaternionic linear group $GL(n, \mathbb{H})$ since ∇^{ob} preserves the hypercomplex structure. The holonomy of the Obata connection is one of the most important invariants on an hypercomplex manifold and it is rarely known explicitly except when an hyperKähler metric exists and the Obata connection coincides with the Levi-Civita connection whose holonomy group is contained in $Sp(n)$. It was shown very recently in [35] that the holonomy of the Obata connection on the Lie group $SU(3)$ coincides with $GL(2, \mathbb{H})$. An important subgroup inside $GL(n, \mathbb{H})$ is its commutator $SL(n, \mathbb{H})$ which appears in the Merkulov-Schwachhöfer list [31] of possible holonomy groups of a torsion-free linear connection. This group can be defined as a group of quaternionic matrices preserving a non-zero complex valued form $\Phi \in \Lambda_{\mathbb{C}}^{2n, 0}(\mathbb{H}_{J_1}^n)$, where $\mathbb{H}_{J_1}^n$ is the n -dimensional quaternionic vector space \mathbb{H}^n considered as a $2n$ -dimensional complex vector space with respect to the complex structure J_1 . A hypercomplex manifold with holonomy of the Obata connection inside $SL(n, \mathbb{H})$ is called *$SL(n, \mathbb{H})$ -manifold*. It was observed by Verbitsky in [40] that $(M, J \in H)$ has holomorphically trivial canonical bundle for any $SL(n, \mathbb{H})$ manifolds (M, H) . For any $SL(n, \mathbb{H})$ -manifold (M, H) and any complex structure $J \in H$ there is a holomorphic volume form $\Phi \in \Lambda^{2n, 0}(M, J)$ with respect to J which is parallel with respect to the Obata connection [40, 5].

For a hypercomplex manifold with holomorphically trivial canonical bundle admitting an HKT metric, a version of Hodge theory was constructed in [39] which leads to the fact established in [40] that a compact hypercomplex manifold with holomorphically trivial canonical bundle is an $SL(n, \mathbb{H})$ manifold if it admits an HKT-structure. Compact simply connected hypercomplex manifold with holomorphically trivial canonical bundle were constructed by A. Swann [38] where it is also shown that some of these examples do not admit any HKT structure.

2.4. The Lee form of an HKT manifold. We recall that the Lee form θ of an almost hermitian structure (g, J) is defined by $\theta = \delta F_J \circ J$, where δ is the co-differential. A hermitian manifold with vanishing Lee form is called *balanced* [32]. For a KT-manifold the Lee form can be expressed in

terms of the torsion as follows [27]

$$(2.6) \quad \theta(X) = -\frac{1}{2} \sum_{i=1}^{2n} T(JX, e_i, Je_i),$$

where e_1, \dots, e_{2n} is an orthonormal basis.

For an HKT manifold, it follows from (2.5) that the torsion three form of an HKT manifold is of type (1,2)+(2,1) with respect to each complex structure $J_s \in H$. It was shown in [24] that in such a case one has the identities

$$(2.7) \quad \sum_{a=1}^{4n} T(J_1 X, e_a, J_1 e_a) = \sum_{a=1}^{4n} T(J_2 X, e_a, J_2 e_a) = \sum_{a=1}^{4n} T(J_3 X, e_a, J_3 e_a).$$

Here and further e_1, \dots, e_{4n} will be an orthonormal basis of TM .

Combining (2.7) with the expression of the Lee form in terms of the torsion, (2.6), we get that the three Lee forms on an HKT-manifold coincide [27] thus obtaining a globally defined one form θ on any HKT manifold defined by (2.6), where $J \in H$ [24, 27, 26]. We call this one form *the Lee form of the HKT manifold*. If the Lee form of an HKT manifold vanishes then we have the notion of a *balanced HKT manifold* (see also [42]).

It was shown in [42] that a balanced HKT manifold is an $SL(n, \mathbb{H})$ manifold but the converse is not true.

The purpose of this note is to find a necessary and sufficient condition an HKT manifold to be an $SL(n, \mathbb{H})$ manifold, i.e. $Hol(\nabla^{ob}) \subset SL(n, \mathbb{H})$. We show that this happens exactly when the Lee form is an exact form.

The aim of the paper is to prove the following

Theorem 2.2. *On an HKT manifold the following conditions are equivalent:*

- a) *The HKT manifold is an $SL(n, \mathbb{H})$ manifold, i.e. $Hol(\nabla^{ob}) \subset SL(n, \mathbb{H})$.*
- b) *The Lee form is an exact form.*

Combining Theorem 2.2 with the already mentioned result of Verbitsky [40], we obtain

Corollary 2.3. *A compact HKT-manifold has holomorphically trivial canonical bundle if and only if the Lee form is exact.*

3. PROOF OF THE MAIN RESULT

First we calculate the difference between the HKT connection and the Obata connection. Surprisingly, we expressed the difference only in terms of the HKT torsion. We have

Proposition 3.1. *On an HKT manifold the Obata connection and the HKT connection are related by*

$$(3.1) \quad \begin{aligned} g(\nabla^{ob}_X Y, Z) &= g(\nabla_X Y, Z) + A(X, Y, Z), \quad \text{where} \\ 2A(X, Y, Z) &= -T(X, J_1 Y, J_1 Z) - T(J_1 X, J_1 Y, Z) - T(X, J_3 Y, J_3 Z) - T(J_1 X, J_3 Y, J_2 Z). \end{aligned}$$

Proof. Obata wrote in [33] a formula connecting the Obata connection with a linear connection with torsion tensor T preserving the hypercomplex structure. Following the proof of [[33], Theorem 10.4] and using the vanishing of the Nijenhuis tensors of a hypercomplex structure, one finds that two connections are related by an (1,2) tensor B having the expression

$$(3.2) \quad \begin{aligned} -4B(X, Y) &= T(X, Y) - J_1 T(X, J_1 Y) - J_2 T(X, J_2 Y) - J_3 T(X, J_3 Y) \\ &\quad + T(J_1 X, J_1 Y) + J_1 T(J_1 X, Y) - J_2 T(J_1 X, J_3 Y) + J_3 T(J_1 X, J_2 Y). \end{aligned}$$

In particular, for the HKT connection we use the special properties of its torsion, namely it is a three form which is of type $(1,2)+(2,1)$ with respect to any complex structure $J \in H$, i.e. the next identities hold

$$(3.3) \quad \begin{aligned} T(X, Y, Z) - T(JX, JY, Z) - T(JX, Y, JZ) - T(X, JY, JZ) &= 0, \quad J \in H; \\ T(J_i X, J_i Y, Z) - T(J_k X, J_k Y, Z) - T(J_k X, J_i Y, J_j Z) - T(J_i X, J_k Y, J_j Z) &= 0, \end{aligned}$$

where $\{i, j, k\}$ is a cyclic permutation of $\{1, 2, 3\}$. Now, using (3.3), we obtain easily that B given in (3.2) is equal to A described with the second equation in (3.1). \square

We need the following important

Lemma 3.2. *On an HKT manifold the difference tensor A between the Obata connection and the HKT connection satisfies the identities*

$$(3.4) \quad \sum_{a=1}^{4n} A(X, e_a, e_a) = -2\theta(X); \quad \sum_{a=1}^{4n} A(X, e_a, J_s e_a) = 0, \quad s = 1, 2, 3.$$

Proof. We calculate from (3.1) applying (2.6), using the quaternionic identities and the fact that the torsion is a three form that

$$(3.5) \quad \begin{aligned} \sum_{a=1}^{4n} A(X, e_a, e_a) &= -\frac{1}{2} \sum_{a=1}^{4n} T(J_1 X, J_1 e_a, e_a) - \frac{1}{2} \sum_{a=1}^{4n} T(J_1 X, J_3 e_a, J_2 e_a) \\ &= -\theta(X) + \frac{1}{2} \sum_{a=1}^{4n} T(J_1 X, e_a, J_1 e_a) = -2\theta(X). \end{aligned}$$

Similarly, we obtain the next sequence of identities

$$(3.6) \quad \sum_{a=1}^{4n} A(X, e_a, J_1 e_a) = \frac{1}{2} \sum_{a=1}^{4n} \left(T(X, J_1 e_a, e_a) - T(X, J_3 e_a, J_2 e_a) \right) = 0;$$

$$(3.7) \quad \begin{aligned} \sum_{a=1}^{4n} A(X, e_a, J_2 e_a) &= -\frac{1}{2} \sum_{a=1}^{4n} \left(T(X, J_1 e_a, J_1 J_2 e_a) + T(J_1 X, J_1 e_a, J_2 e_a) \right) \\ &\quad - \frac{1}{2} \sum_{a=1}^{4n} \left(T(X, J_3 e_a, J_3 J_2 e_a) - T(J_1 X, J_3 e_a, e_a) \right) = \sum_{a=1}^{4n} \left(T(X, e_a, J_2 e_a) - T(J_1 X, e_a, J_3 e_a) \right) \\ &= 2\theta(J_2 X) - 2\theta(J_3 J_1 X) = 0; \end{aligned}$$

$$(3.8) \quad \begin{aligned} \sum_{a=1}^{4n} A(X, e_a, J_3 e_a) &= -\frac{1}{2} \sum_{a=1}^{4n} \left(T(X, J_1 e_a, J_1 J_3 e_a) + T(J_1 X, J_1 e_a, J_3 e_a) \right) \\ &\quad + \frac{1}{2} \sum_{a=1}^{4n} \left(T(X, J_3 e_a, e_a) - T(J_1 X, J_3 e_a, J_2 J_3 e_a) \right) = 0. \end{aligned}$$

Now, (3.4) follow from (3.5), (3.6), (3.7) and (3.8) which completes the proof of Lemma 3.2. \square

3.1. Proof of Theorem 2.2. We recall that the non-degenerate 2-form $\Omega_i = F_j + \sqrt{-1}F_k$ is of type (2,0) with respect to the complex structure J_i and it is parallel with respect to the HKT-connection, $\nabla\Omega_i = 0$. Hence, the (2n,0)-form Ω_i^n is non-degenerate complex volume form which is ∇ -parallel, $\nabla\Omega_i^n = 0$.

Let the Lee form of the HKT-structure be an exact form, $\theta = df$. We claim that the (2n,0)-form $\Phi = e^{-2f}\Omega_i^n$ is parallel with respect to the Obata connection.

Let $e_1, \dots, e_{2n}, J_i e_1, \dots, J_i e_{2n}$ be an orthonormal basis of TM and $E_\alpha = e_\alpha - \sqrt{-1}J_i e_\alpha$, $\alpha = 1, \dots, 2n$ be a basis of the (1,0)-space $T_{J_i}^{1,0}M$ with respect to J_i . The complex conjugate of E_α is as usual $\overline{E}_\alpha = e_\alpha + \sqrt{-1}J_i e_\alpha$.

For a real vector X , we calculate from (3.1) that

$$(3.9) \quad (\nabla^{ob}_X \Phi)(E_1, \dots, E_{2n}) = (\nabla_X \Phi)(E_1, \dots, E_{2n}) - \sum_{\alpha=1}^{2n} A(X, E_\alpha, \overline{E}_\alpha) \Phi(E_1, \dots, E_{2n}) \\ = -2df(X)\Phi(E_1, \dots, E_{2n}) + e^{-2f}(\nabla_X \Omega_i^n)(E_1, \dots, E_{2n}) + 2\theta(X)\Phi(E_1, \dots, E_{2n}) = 0,$$

since $\nabla\Omega_i^n = 0$, $\theta = df$ and the identity $\sum_{\alpha=1}^{2n} A(X, E_\alpha, \overline{E}_\alpha) = -2\theta(X)$. To see the latter, we calculate using Lemma 3.2 that

$$(3.10) \quad \sum_{\alpha=1}^{2n} A(X, E_\alpha, \overline{E}_\alpha) \\ = \sum_{a=1}^{2n} \left(A(X, e_a, e_a) + A(X, J_i e_a, J_i e_a) + \sqrt{-1} \left[A(X, e_a, J_i e_a) - A(X, J_i e_a, e_a) \right] \right) \\ = -2\theta(X).$$

For the converse, suppose that there exist a (2n,0)-form Ψ which is parallel with respect to the Obata connection, $\nabla^{ob}\Psi = 0$. Hence, $|\Psi|^2 > 0$. We have similarly as above that

$$(3.11) \quad 0 = (\nabla^{ob}_X \Psi)(E_1, \dots, E_{2n}) = (\nabla_X \Psi)(E_1, \dots, E_{2n}) - \sum_{\alpha=1}^{2n} A(X, E_\alpha, \overline{E}_\alpha) \Psi(E_1, \dots, E_{2n}).$$

Apply (3.10) to (3.11) to conclude

$$(3.12) \quad (\nabla_X \Psi)(E_1, \dots, E_{2n}) = -2\theta(X)\Psi(E_1, \dots, E_{2n}).$$

The identity (3.12) yields

$$\theta = -\frac{1}{4}d(\ln|\Psi|^2)$$

since the HKT-connection preserves the hyperhermitian structure.

Thus, the proof of Theorem 2.2 is completed.

4. CURVATURE OF THE OBATA CONNECTION

Let (M, g, H, ∇) be a 4n-dimensional HKT manifold. Let $R = [\nabla, \nabla] - \nabla_{[\cdot, \cdot]}$ be the curvature tensor of ∇ and R^{ob}, R^g be the curvature of the Obata and the Levi-Civita connection, respectively. Further we used the superscript ob , (resp. g) to denote tensors obtained from the Obata connection ∇^{ob} (resp. obtained from the Levi-Civita connection ∇^g). We denote the curvature tensor of type (0,4) by the same letter, $R(X, Y, Z, U) := g(R(X, Y)Z, U)$. Note that R^{ob} is not skew-symmetric with respect to the second pair of arguments since ∇^{ob} is not a metric connection.

The Ricci tensor Ric , the Ricc-type 2-forms ρ, ρ_s and the scalar curvatures $Scal, Scal_s$ are defined as follows

$$\begin{aligned} Ric(X, Y) &= \sum_{a=1}^{4n} R(e_a, X, Y, e_a) \quad Scal = \sum_{a=1}^{4n} Ric(e_a, e_a), \quad Scal_s = \sum_{a=1}^{4n} Ric(J_s e_a, e_a), \\ \rho(X, Y) &= \sum_{a=1}^{4n} R(X, Y, e_a, e_a), \quad \rho_s(X, Y) = \frac{1}{2} \sum_{a=1}^{4n} R(X, Y, e_a, I_s e_a), \quad s = 1, 2, 3. \end{aligned}$$

We have

Proposition 4.1. *On a hypercomplex manifold (M, H) , for $s = 1, 2, 3$, we have*

$$(4.1) \quad Ric^{ob}(J_s X, J_s Y) + Ric^{ob}(Y, X) = 2\rho_s^{ob}(J_s X, Y), \quad Ric^{ob}(X, Y) - Ric^{ob}(Y, X) = -\rho^{ob}(X, Y).$$

On an HKT manifold we have:

- a) *The exterior derivative of the Lee form of an HKT manifold is an $(1,1)$ form with respect to the hypercomplex structure,*

$$d\theta(J_s X, J_s Y) = d\theta(X, Y), \quad s = 1, 2, 3;$$

- b) *The Ricci tensor of the Obata connection of an HKT manifold is skew-symmetric determined by the Lee form and we have the identities*

$$(4.2) \quad Ric^{ob}(X, Y) = d\theta(X, Y), \quad \rho^{ob} = -2d\theta, \quad \rho_s^{ob} = 0, \quad s = 1, 2, 3.$$

In particular, the Ricci tensor of the Obata connection of an HKT manifold is an $(1,1)$ form with respect to the hypercomplex structure, $Ric^{ob}(J_s X, J_s Y) = Ric^{ob}(X, Y)$.

- c) *The scalar curvatures of the Obata connection vanish,*

$$Scal^{ob} = Scal_s^{ob} = 0, \quad s = 1, 2, 3.$$

Proof. Let g be a Riemannian metric hermitian compatible with the hypercomplex structure. Such a metric always exists. For example, take any Riemannian metric h then the metric $g(X, Y) = h(X, Y) + \sum_{s=1}^3 h(J_s X, J_s Y)$ is hyperhermitian. The first Bianchi identity and the conditions $R^{ob} J_s = J_s R^{ob}$, $s = 1, 2, 3$ imply the following sequence of identities

$$\begin{aligned} Ric^{ob}(X, Y) &= - \sum_{a=1}^{4n} \left(R^{ob}(X, Y, e_a, e_a) + R^{ob}(Y, e_a, X, e_a) \right) = -\rho^{ob}(X, Y) + Ric^{ob}(Y, X); \\ 2\rho_s^{ob}(X, Y) &= \sum_{a=1}^{4n} R^{ob}(X, Y, e_a, J_s e_a) = \sum_{a=1}^{4n} \left(R^{ob}(Y, e_a, J_s X, e_a) + R^{ob}(e_a, X, J_s Y, e_a) \right) \\ &= -Ric^{ob}(Y, J_s X) + Ric^{ob}(X, J_s Y), \quad s = 1, 2, 3 \end{aligned}$$

which proves (4.1).

Now, let (M, g, H, ∇) be an HKT manifold. Using (3.1) we obtain after standard calculations that the curvature of the Obata connection and the HKT-connection are related by

$$(4.3) \quad \begin{aligned} R^{ob}(X, Y, Z, U) &= R(X, Y, Z, U) + (\nabla_X A)(Y, Z, U) - (\nabla_Y A)(X, Z, U) \\ &\quad + A(T(X, Y), Z, U) + A(X, A(Y, Z), U) - A(Y, A(X, Z), U), \end{aligned}$$

where A is given by the second equation in (3.1).

Further, since the HKT-connection preserves the hyperhermitian structure its holonomy is contained in $Sp(n)$ and we have

$$(4.4) \quad \rho = \rho_s = 0, \quad s = 1, 2, 3.$$

Taking the traces in (4.3) and using (3.4) and (4.4), we obtain

$$(4.5) \quad \rho^{ob}(X, Y) = -2(\nabla_X \theta)Y + 2(\nabla_Y \theta)X - 2\theta(T(X, Y)) \\ + \sum_{a,b=1}^{4n} \left(A(X, e_b, e_a)A(Y, e_a, e_b) - A(Y, e_b, e_a)A(X, e_a, e_b) \right) = -2d\theta(X, Y),$$

where we used the expression of the exterior derivative of an one form α with respect to a metric connection with torsion T , $d\alpha(X, Y) = (\nabla_X \alpha)Y - (\nabla_Y \alpha)X + \alpha(T(X, Y))$.

We calculate from (4.3), using (3.4) and (4.4) that

$$(4.6) \quad \rho_s^{ob}(X, Y) = \sum_{a,b=1}^{4n} \left(A(X, e_b, J_s e_a)A(Y, e_a, e_b) - A(Y, e_b, J_s e_a)A(X, e_a, e_b) \right) \\ = \sum_{a,b=1}^{4n} \left(A(Y, e_a, J_s e_b) \left[A(X, J_s e_b, J_s e_a) - A(X, e_b, e_a) \right] \right) = 0, \quad s = 1, 2, 3.$$

The last equality in (4.6) follows from the identity $A(X, J_s Y, J_s Z) - A(X, Y, Z) = 0$, $s = 1, 2, 3$ which is a consequence from the fact that both the Obata and the HKT-connections preserve the hyperhermitian structure.

The second and the third equality in (4.2) follow from (4.5) and (4.6).

Using (4.6), we obtain from (4.1) that

$$(4.7) \quad Ric^{ob}(J_s X, J_s Y) + Ric^{ob}(Y, X) = 0, \quad s = 1, 2, 3 \quad \text{yielding} \\ Ric^{ob}(J_s X, J_s Y) = Ric^{ob}(J_t X, J_t Y) = Ric^{ob}(X, Y), \quad s, t = 1, 2, 3.$$

The two equalities (4.7) lead to $Ric^{ob}(X, Y) + Ric^{ob}(Y, X) = 0$ which combined with the second equality in (4.1) and (4.5) imply the first equality in (4.2). This proves b).

The condition a) and $Scal^{ob} = 0$ follow from the second equality in (4.7) and the first equality in (4.2).

To complete the proof of c) we have to show that $d\theta$ is completely trace-free. Fix $s \in \{1, 2, 3\}$ and consider the 1-form $J_s \theta$ defined by $J_s \theta(X) = -\theta(J_s X)$. The condition $\theta = \delta F_s \circ J_s$ implies $J_s = \delta F_s$ and in particular the 1-form $J_s \theta$ is co-closed, $\delta(J_s \theta) = 0$. Expressing the latter in terms of ∇^g and ∇ , we get

$$0 = \delta(J_s \theta) = - \sum_{a=1}^{4n} (\nabla_{e_a}^g J_s \theta)(e_a) = - \sum_{a=1}^{4n} (\nabla_{e_a} J_s \theta)(e_a) = \sum_{a=1}^{4n} (\nabla_{e_a} \theta)(J_s e_a)$$

where the third equality follows from $\nabla^g = \nabla - \frac{1}{2}T$ and the fourth equality is a consequence of $\nabla J_s = 0$. Then we have

$$\sum_{a=1}^{4n} d\theta(e_a, J_s e_a) = \sum_{a=1}^{4n} \left[2(\nabla_{e_a} \theta)(J_s e_a) + \theta(T(e_a, J_s e_a)) \right] = g(\theta, J_s \theta) = 0$$

where we used the expression of $d\theta$ in terms of the torsion connection ∇ and the definition of the Lee form (2.6). \square

It is known from [1] that the restricted holonomy group of the Obata connection on an hypercomplex manifold is a subgroup of $SL(n, \mathbb{H})$ if and only if its Ricci tensor vanishes, $Ric^{ob} = 0$. On the other hand we have the inclusions $SL(n, \mathbb{H}) \subset SL(2n, \mathbb{C}) \subset SL(4n, \mathbb{R})$ which shows that $Hol(\nabla^{ob}) \subset SL(n, \mathbb{H})$ exactly when all Ricci two forms of the Obata connection vanish. We obtain from Proposition 4.1 the next

Corollary 4.2. *The restricted holonomy group of the Obata connection on an HKT manifold is contained in $SL(n, \mathbb{H})$ if and only if the Lee form is closed, $d\theta = 0$;*

Note that Corollary 4.2 also follows from Theorem 2.2.

We remark that if the Lee form is closed but not exact the restricted holonomy group of the Obata connection is contained in $SL(n, \mathbb{H})$ but the whole holonomy group of the Obata connection may not be contained in $SL(n, \mathbb{H})$. As pointed out in [42] this happens in the case of Hopf manifolds $(\mathbb{H}^n - \{0\})/\Gamma$ which have flat Obata connection but do not admit a holomorphic volume form and therefore these HKT manifolds are not $SL(n, \mathbb{H})$ manifolds.

4.1. Non existence of HKT metric. As a direct consequence of Proposition 4.1 one gets a simple criterion for non-existence of HKT metric in terms of the Ricci-type tensors of the Obata connection. Comparing the statements in Proposition 4.1, we obtain

Corollary 4.3. *Let (M, H) be a hypercomplex manifold. Then there is no HKT structure on M compatible with the hypercomplex structure H if any of the following three conditions hold:*

- a) *The Ricci tensor of the Obata connection is either not skew-symmetric or not $(1,1)$ -form with respect to the hypercomplex structure;*
- b) *At least one of the Ricci-forms of the Obata connection does not vanish identically, $\rho_s^{ob} \neq 0$ for some $s \in \{1, 2, 3\}$;*
- c) *At least one of the scalar curvatures of the Obata connection is different from zero.*

We remark that if the conditions a), b) and c) of Corollary 4.3 are satisfied then this does not imply the existence of HKT structure due to the compact examples presented by A. Swann [38] of $SL(n, \mathbb{H})$ manifolds which have vanishing Ricci tensor of the Obata connection [1] but do not admit any HKT structure.

5. HYPERKÄHLER HKT SPACES

In this section we give sufficient conditions a compact HKT manifold to be hyperKähler in terms of the traces of the exterior derivative of the torsion and the *-scalar curvature of the Levi-Civita connection.

It was observed in [3] that a strong balanced KT manifold is Kähler (see [10] for a different independent proof). More general, the equality (2.13) in [3] written in the form (see [27, (3.10)]),

$$(5.1) \quad \sum_{a,b=1}^{2n} dT(e_a, Je_a, e_b, Je_b) = 8\delta\theta + 8|\theta|^2 - \frac{4}{3}|T|^2,$$

shows that a balanced KT manifold satisfying $\sum_{a,b=1}^{2n} dT(e_a, Je_a, e_b, Je_b) = 0$ is Kähler. If the manifold is compact, a particular case of the vanishing theorem [27, Theorem 4.1], [28] states

Theorem 5.1. [27] *A compact (non Kähler) KT manifold with restricted holonomy of the KT-connection contained in $SU(n)$ satisfying the condition $\sum_{a,b=1}^{2n} dT(e_a, Je_a, e_b, Je_b) = 0$ admits no holomorphic volume form.*

We recall the slightly general notion of a QKT manifold which is defined as a quaternionic hermitian manifold of dimension $4n > 4$ admitting a linear connection preserving the quaternionic structure and having totally skew-symmetric torsion which is an $(1,2)+(2,1)$ three form with respect to the quaternionic structure, the notion investigated by Howe, Opfermann and Papadopoulos [20] in connection with supersymmetric sigma models with Wess-Zumino term. We note that an HKT manifold is always a QKT manifold. The QKT-torsion 1-form t defined in [24] coincides (up to a sign) with the Lee form θ in case of an HKT space.

For a fixed $s \in \{1, 2, 3\}$, the scalar curvature $Scal_s^g$ of the Levi-Civita connection ∇^g is also known as $*$ -scalar curvature considering (M, g, J_s) as an almost hermitian manifold. The $*$ -scalar curvature is also equal to $Scal_s^g = \sum_{a=1}^{4n} \rho_s^g(J_s e_a, e_a)$ due to the first Bianchi identity for R^g .

Since any HKT space is a QKT manifold, it follows from [26, Proposition 3.4, Proposition 3.1] that the three $*$ -scalar curvatures on an HKT manifold of dimension grater then four coincide, $Scal_1^g = Scal_2^g = Scal_3^g$, and the common scalar curvature $Scal_H^g = Scal_1^g$, called *the $*$ -scalar curvature of an HKT manifold*, is given by

$$(5.2) \quad Scal_H^g = \frac{1}{8} \sum_{a,b=1}^{4n} dT(e_a, J_1 e_a, e_b, J_1 e_b) + \frac{1}{12} |T|^2$$

because the Ricci forms of the HKT connection ∇ vanish, $\rho_s = 0, s = 1, 2, 3$. In fact, the proof of [26, Proposition 3.4] shows that the above conclusions hold also in dimension four. Indeed, for any HKT manifold, the formula (3.12) in [26], taken with $t = -\theta$, reads

$$\rho^g(X, J_s Y) = \frac{1}{2} (\nabla_X \theta) Y + \frac{1}{2} (\nabla_{J_s Y} \theta) J_s X - \frac{1}{2} \theta (J_s T(X, J_s Y)) + \frac{1}{4} \sum_{a,b=1}^{4n} T(X, e_a, e_b) T(J_s Y, J_s e_a, e_b)$$

for $s = 1, 2, 3$, where we used $\rho_s = 0$ for an HKT manifold. The trace of the above equality with an application of [26, Lemma 3.2] and (2.6) gives

$$(5.3) \quad Scal_s^g = \delta\theta + |\theta|^2 - \frac{1}{12} |T|^2 = \frac{1}{8} \sum_{a,b=1}^{4n} dT(e_a, J_s e_a, e_b, J_s e_b) + \frac{1}{12} |T|^2, \quad s = 1, 2, 3,$$

where we applied (5.1) to obtain the second equality of (5.3).

We derive from Theorem 2.2 and the vanishing results in [27, 28] the next

Theorem 5.2. *A compact HKT manifold with an exact Lee form is hyperKähler if any of the following two conditions hold*

- 1). *The function $h = -\frac{1}{4} \sum_{a,b=1}^{4n} dT(e_a, J_1 e_a, e_b, J_1 e_b)$ vanishes identically, $h = 0$;*
- 2). *the $*$ -scalar curvature is zero, $Scal_H^g = 0$.*

Proof. We apply the vanishing results from [28, 27] to show that a compact non hyperKähler HKT manifold satisfying any of the conditions 1), 2) of the theorem admits no holomorphic volume form, a contradiction with Corollary 2.3.

Since the holonomy of the torsion connection ∇ of an HKT manifold is contained in $Sp(n) \subset SU(2n)$ we apply [27, Corollary 4.2 (b)]. Corollary 4.2 (b) in [27] states that if the function $|C|^2 - h$, where C is the torsion of the Chern connection of a KT manifold (M, g, J) , is strictly positive then $p_m(J) = \dim H^0(M, \mathcal{O}(K^m))$ vanish for $m > 0$ and, in particular, the complex manifold (M, J) admits no holomorphic volume form.

We recall that the torsion C of the Chern connection of a KT manifold (M, g, J) is expressed in terms of the torsion three form T as follows, see e.g. [27]

$$g(C(X, Y), Z) = \frac{1}{2} T(X, JY, JZ) + \frac{1}{2} T(JX, Y, JZ).$$

The last equality together with an application of [26, Lemma 3.2] implies

$$(5.4) \quad |C_1|^2 = |C_2|^2 = |C_3|^2 = \frac{1}{3} |T|^2,$$

where C_1, C_2, C_3 denote the torsion of the Chern connections of $(M, g, J_1), (M, g, J_2), (M, g, J_3)$, respectively.

If $h = 0$ then clearly $|C_s|^2 - h = |C_s|^2 = \frac{1}{3} |T|^2, s = 1, 2, 3$, where we applied (5.4).

Further, the condition $Scal_H^g = 0$ together with (5.2) gives $h = \frac{1}{6}|T|^2$. Using (5.4) we obtain $|C_s|^2 - h = \frac{1}{6}|T|^2$, $s = 1, 2, 3$.

Hence, in both cases of the conditions of the theorem each of the functions $|C_s|^2 - h$, $s = 1, 2, 3$ is a positive multiple of $|T|^2$ and therefore it is strictly positive if the HKT space is not hyperKähler. Now, [27, Corollary 4.2 (b)] shows that the non hyperKähler HKT manifold admits no holomorphic volume form which contradicts Corollary 2.3 since the Lee form is exact. \square

For an HKT manifold the exterior derivative dT of the torsion three form is of type $(2, 2)$ with respect to the hypercomplex structure H and, as shown in [24], we have the equalities $\sum_{a=1}^{4n} dT(e_a, J_1 e_a, X, J_1 Y) = \sum_{a=1}^{4n} dT(e_a, J_2 e_a, X, J_2 Y) = \sum_{a=1}^{4n} dT(e_a, J_3 e_a, X, J_3 Y)$ which allows us to define an *almost strong HKT manifold* as an HKT manifold satisfying the condition $\sum_{a=1}^{4n} dT(e_a, J_1 e_a, X, J_1 Y) = 0$. Theorem 5.2 yields the following

Corollary 5.3. *A compact almost strong HKT manifold with an exact Lee form is hyperKähler.*

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